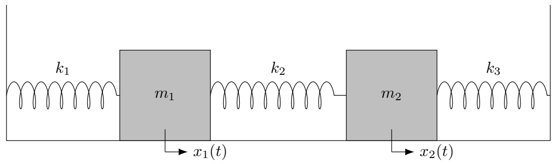
Intro. to ODEs

Quiz 12 **Solutions**

1) Consider the coupled spring–mass system shown below. Take $m_1 = m_2 = 2$, $k_1 = k_3 = 2$, and $k_2 = 3$. You can ignore resistance for this problem.




a) Find a $4 \times 4$ matrix $M$ so that the motion of the two masses can be modeled by

$$\begin{bmatrix} x_1' \\ x_2' \\ v_1' \\ v_2' \end{bmatrix} = M \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix}.$$

$$2x_1'' = -2x_1 + 3(x_2 - x_1) = -5x_1 + 3x_2$$
$$2x_2'' = -3(x_2 - x_1) - 2x_2 = 3x_1 - -5x_2$$

$$\begin{bmatrix} x_1' \\ x_2' \\ v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5/2 & 3/2 & 0 & 0 \\ 3/2 & -5/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix}.$$

b) If your matrix $M$ is correct, $\lambda = i$ should be an eigenvalue. Find a corresponding eigenvector for this eigenvalue.

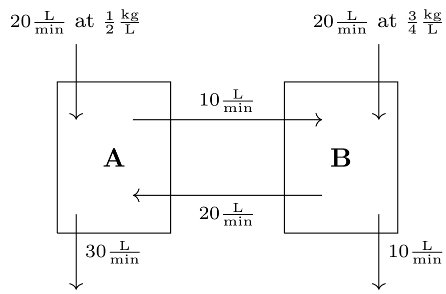
$$\left[\begin{array}{cccc|c} -i & 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 1 & 0 \\ -5/2 & 3/2 & -i & 0 & 0 \\ 3/2 & -5/2 & 0 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & i & 0 \\ 0 & 1 & 0 & i & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This means we need a vector $\vec{v}$ satisfying $v_1 = -iv_4$, $v_2 = -iv_4$, and $v_3 = v_4$ with $v_4$ free. Choosing $v_4 = -1$ gives us one such eigenvector.

$$\vec{v} = \begin{bmatrix} i \\ i \\ -1 \\ -1 \end{bmatrix}$$

Any (complex) multiple of this vector would also work!

2) Two tanks are being used to prepare mixtures of a salt solution. Both tanks are being held at a constant volume of 100 L with the flow rates as shown in the diagram below.




a) Set-up a system of inhomogeneous first-order linear differential equations that describes the total amounts of salt in tanks **A** and **B** at time $t$. Use $x(t)$ for the total amount of salt in tank **A**, and $y(t)$ for the total amount of salt in tank **B**.

b) Find the particular solution to this inhomogeneous system. You DO NOT need to find the homogeneous solution.

a)

$$\frac{dx}{dt} = 10 - 40\left(\frac{x}{100}\right) + 20\left(\frac{y}{100}\right) = -\frac{2}{5}x + \frac{1}{5}y + 10$$
$$\frac{dy}{dt} = 15 + 10\left(\frac{x}{100}\right) - 30\left(\frac{y}{100}\right) = \frac{1}{10}x - \frac{3}{10}y + 15$$

In vector–form, the system is modeled by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2/5 & 1/5 \\ 1/10 & -3/10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 10 \\ 15 \end{bmatrix}.$$

b) Since the inhomogeneous term is a constant, we guess a particular solution of the form

$$\vec{x_p}(t) = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Running this through the ODE gives

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/5 & 1/5 \\ 1/10 & -3/10 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 10 \\ 15 \end{bmatrix},$$

which is equivalent to the system of equations

$$2a - b = 50$$
$$-a + 3b = 150.$$

The solution to this system is $a = 60, b = 70$ which gives a particular solution of

$$\vec{x_p}(t) = \begin{bmatrix} 60 \\ 70 \end{bmatrix}.$$

Note that this means the *concentration* in tank **A** will limit to $3/5$ kg/L while the concentration in tank **B** will limit to $7/10$ kg/L.